Note on a Physical Meaning of Connection Coefficients of the Cartan Space associated with Compressible Fluid Flows

Takeyuki UEHARA

Department of Mechanical Engineering, Faculty of Engineering

1. Introduction

As has already been shown in (1), (2) and (3), the motion of a compressible fluid is well represented in the Cartan space based on the concept of area, and the characteristic features of the fluid motion have been clarified in geometrical terminology. However it has been left as a pending problem to ask the physical counterpart of the coefficients of connection which are important structural characteristics of the Cartan space. We shall begin with the study of Earnshaw paradox and then show that the coefficients of connection of our Cartan space are interpreted as the distortion of the plane sound wave of finite amplitude.

2. The Euclidean connection of the Cartan space

We have already introduced five-dimensional Cartan space of which the element is given by $(x^1, x^2, x^3, t, \Phi, q_1, q_2, q_3, \phi, 1)$, as the underlying space of the three-dimensional irrotational unsteady motion of a compressible fluid satisfying the isentropy condition, where x^1, x^2 and x^3 are the ordinary three-dimensional Cartesian coordinates, t the time variable, Φ the four-dimensional velocity potential defined by

$$\left.\begin{aligned} q_i&=\frac{\partial\Phi}{\partial x^i}, \quad (i=1,\ 2,\ 3)\\ \phi&=\frac{\partial\Phi}{\partial t}, \end{aligned}\right\} \tag{2. 1}$$

q_1, q_2, q_3 the velocity components along the x^1-, x^2- and x^3-axis respectively and ϕ is the negative of the total energy per unit mass. Further we have determined the metric of the space by means of the formlae

$$\left.\begin{aligned} &gg^{\kappa\lambda}=\frac{1}{2}\frac{\partial^2(L^2)}{\partial u_\lambda \partial u_\lambda}\\ &g^{\kappa\mu}g_{\mu\lambda}=\delta^\kappa_\lambda, \quad g=\det\ (g_{\lambda\kappa})\\ &\qquad\qquad (\kappa,\ \lambda,\ \mu=0,\ 1,\ 2,\ 3,\ 4)\\ &L(x^\lambda,\ u_\lambda)=u_0 p(W),\\ &W=-\frac{1}{2}\frac{u_1{}^2+u_2{}^2+u_3{}^2}{u_0{}^2}+\frac{u^4}{u_0}-\Omega(x^1, x^2, x^3, x^4) \end{aligned}\right\} \tag{2. 2}$$

where $p(W)$ is the pressure as the function of W and Ω the potential of the external force, and

$$\left.\begin{array}{l} x^0=\Phi\ ,\ \ x^4=t, \\ -\dfrac{u_1}{u_0}=q_1,\ \ -\dfrac{u_2}{u_0}=q_2,\ \ -\dfrac{u_3}{u_0}=q_3,\ \ -\dfrac{u_4}{u_0}=\phi\ . \end{array}\right\} \tag{2.3}$$

Thus the metric tensor of our underlying Cartan space have been able to be represented in terms of physical quantities which appear in the fluid motion such as the velocity, the pressure, the density and the local velocity of sound etc.* The potential equation for a compressible fluid flow has also been represented as that of a hypersurface in our Cartan space.**

Now, according to E.Cartan the Euclidean connection of our space is given by in the form***

$$DX^\kappa=dX^\kappa+X^\mu C^{\lambda\kappa}_{\mu}du_\lambda+X^\mu\Gamma_{\mu\lambda}^{\kappa}dx^\lambda \tag{2.4}$$

where DX^κ is the absolute differential of a contravariant vector X^κ on the element $(x^\kappa,\ u_\lambda)$, and both $C^{\lambda\kappa}_{\mu}$ and $\Gamma^\kappa{}_{\mu\lambda}$ are the coefficients of connection. However in our problem, the latter is not so significant as the former because of the fact that we consider on the laminar flow in the Eulerian viewpoint where the velocity of fluid is treated as a function of space and time vatiables. So we confine ourselves in this note to the coefficients of conection $C^{\lambda\kappa}_{\mu}$ which are defined by

$$C^{\mu\lambda\kappa}=-\frac{1}{2}\frac{\partial g^{\mu\lambda}}{\partial u_\kappa}\ . \tag{2.5}$$

3. The Condition of propagation of Plane Sound Wave in a Compressible Fluid

At first we shall show in an elementary way that plane sound waves of finite amplitude can propargate through a compressible invicid fluid without torsion if and only if the fluid satisfies an equation of state of the special form. We assume that the form of the sound wave is stationary, and that the train of waves moves with constant velocity normal to the wave fronts. If we adopt a coordinate system moving with the waves, the fluid motion will be not only one-dimensional but also steady. Hence we can write

$$\left.\begin{array}{ll} \rho=\rho(x), & p=p(x), \\ u=u(x), & \end{array}\right\} \tag{3.1}$$

where x is the coordinate in the direction of wave motion, ρ denotes the density, p the pressure and u the velocity component along the x-axis.

Neglecting gravity, Bernoulli's equation reduces in our case to

$$udu+\frac{dp}{\rho}=0, \tag{3.2}$$

and the equation of continuity reduces to

$$\rho u=\text{const.}=C\ \text{ or }\ u=\frac{C}{\rho} \tag{3.3}$$

* As for the detail, see (2) on pp. 8～25.

** See (1).

*** See(4) on p.14.

Substituting (3.3) into (3.2) we get

$$-\frac{C^2 d\rho}{\rho^3}+\frac{dp}{\rho}=0 \text{ or } dp=\frac{C^2 d\rho}{\rho^2}\cdot \tag{3.4}$$

After integration we have

$$p=A-\frac{C^2}{\rho}, \tag{3.5}$$

where A is a constant.

Hence such a wave motion mentioned above is possible if and only if the fluid satisfies the equation of the special form (3.5) which is sometimes called the Earnshaw-type relation between the pressure and the density.

Obviously (3.5) is never included in the equation of state of polytropic gases

$$p=\alpha\rho^{\nu} \tag{3.6}$$

which is generally admitted as valid in most cases in high-speed flows of a compressible invicid fluid, where ν is the specific heat ratio and α is a constant.

G.Birkhoff treated this contradiction as a typical paradox due to overdeterminism and called it the Earnshaw paradox* but we may regard (3.5) as the condition that plane sound waves of finite amplitude and stationary form, if exist, can propagate whitout distorsion.

Now consider the same problem from the geometrical viewpoint. As was shown in (2), the most general potential equation for a compressible fluid flow is given by using the metric tensor of our Cartan space as follows:

$$(g^{\kappa\lambda}g^{\mu\nu}-g^{\kappa\mu}g^{\lambda\nu})u_\nu u_\mu \partial_\lambda u_\kappa=0, \quad (\kappa,\ \lambda,\ \mu,\ \nu=0,\ 1,\ 2,\ 3,\ 4), \tag{3.7}$$

where $g^{\kappa\lambda}$ and u_λ is defined by (2.2) and (2.3), and

$$\partial_\lambda=\frac{\partial}{\partial x^\lambda}\cdot$$

Separating the time variable, we get

$$(g^{44}g^{\mu\nu}-g^{4\mu}g^{4\nu})u_\nu u_\mu \partial_4 u_4+2(g^{4i}g^{\mu\nu}-g^{4\mu}g^{i\nu})u_\nu u_\mu \partial_i u_4$$
$$=-(g^{ij}g^{\mu\nu}-g^{i\mu}g^{j\nu})u_\nu u_\mu \partial_j u_i, \ (\mu,\ \nu=0,1,2,3,4),\ (i,j=0,1,2,3) \tag{3.8}$$

or more explicitely

$$(g^{44}g^{\mu\nu}-g^{4\mu}g^{4\nu})q_\nu q_\mu \frac{\partial^2\Phi}{\partial t^2}+2(g^{4i}g^{\mu\nu}-g^{4\mu}g^{i\nu})q_\nu q_\mu \frac{\partial}{\partial x^i}\ \frac{\partial\Phi}{\partial t}$$
$$=-(g^{ij}g^{\mu\nu}-g^{i\mu}g^{j\nu})q_\nu q_\mu \frac{\partial^2\Phi}{\partial x^j \partial x^i}\cdot \tag{3.8'}$$

If we adopt a three dimensional coordinate system moving with plane waves, the fluid motion will be regarded as steady. Thus

$$\frac{\partial\Phi}{\partial t}=0$$

holds in (3.8'). Hence we obtain

$$(g^{i\mu}g^{j\nu}-g^{ij}g^{\mu\nu})q_\nu q_\mu \frac{\partial^2\Phi}{\partial x^j \partial x^i}=0, \quad (\mu,\nu=0,1,2,3,4), \quad (i,j=0,1,2,3) \tag{3.9}$$

---

* See (5) on pp. 22~23.

for the necessary and sufficient condition that plane sound waves of finite amplitude can propargate through an invicid fluid whitout distorsion.

Now consider the case when the fluid velocity relative to moving waves is small enough to neglect terms of higher order. If we put

$$g^{i\mu}g^{j\nu}-g^{ij}g^{\mu\nu}\equiv G^{ij\mu\nu}, \tag{3.10}$$

then on account of (2.5) we have

$$\begin{aligned}\frac{\partial G^{ij\mu\nu}}{\partial u_\sigma}&=\frac{\partial g^{i\mu}}{\partial u_\sigma}g^{j\nu}+g^{i\mu}\frac{\partial g^{j\nu}}{\partial u_\sigma}-\frac{\partial g^{ij}}{\partial u_\sigma}g^{\mu\nu}-g^{ij}\frac{\partial g^{\mu\nu}}{\partial u_\sigma}\\&=2\left(C^{ij\sigma}g^{\mu\nu}+C^{\mu\nu\sigma}g^{ij}-C^{i\mu\sigma}g^{j\nu}-C^{j\nu\sigma}g^{i\mu}\right)\text{ .}\end{aligned} \tag{3.11}$$

Expanding $G^{ij\mu\nu}$ in terms of the relative veloccity, we get

$$\begin{aligned}G^{ij\mu\nu}=&\overset{*}{G}{}^{ij\mu\nu}+2\left(\overset{*}{C}{}^{ij\sigma}\overset{*}{g}{}^{\mu\nu}+\overset{*}{C}{}^{\mu\nu\sigma}\overset{*}{g}{}^{ij}-\overset{*}{C}{}^{i\mu\sigma}\overset{*}{g}{}^{j\nu}-\overset{*}{C}{}^{i\nu\sigma}\overset{*}{g}{}^{i\mu}\right)(u_\sigma-\overset{*}{u}_\sigma)\\&+0\{(u_\sigma-\overset{*}{u}_\sigma)^2\}\end{aligned} \tag{3.12}$$

from (3.11), where * is put on quantities in reference to moving waves. Therefore the condition (3.9) is now written in the form

$$\begin{aligned}&[\overset{*}{G}{}^{ij\mu\nu}+2u_0\left(\overset{*}{C}{}^{ij\sigma}\overset{*}{g}{}^{\mu\nu}+\overset{*}{C}{}^{\mu\nu\sigma}\overset{*}{g}{}^{ij}-\overset{*}{C}{}^{i\mu\sigma}\overset{*}{g}{}^{j\nu}-\overset{*}{C}{}^{j\nu\sigma}\overset{*}{g}{}^{i\mu}\right)(q_\sigma-\overset{*}{q}_\sigma)\\&\quad+0\{(q_\sigma-\overset{*}{q}_\sigma)^2\}]q_\nu q_\mu\frac{\partial^2\Phi}{\partial x^j\partial x^i}=0\text{ .}\end{aligned} \tag{3.13}$$

In order that the equation (3.12) holds for the arbitrary q_λ

$$\overset{*}{G}{}^{ii\mu\nu}=0\ , \tag{3.14}$$

and

$$\begin{aligned}&\overset{*}{C}{}^{ij\sigma}\overset{*}{g}{}^{\mu\nu}+\overset{*}{C}{}^{\mu\nu\sigma}\overset{*}{g}{}^{ij}-\overset{*}{C}{}^{i\mu\sigma}\overset{*}{g}{}^{j\nu}-\overset{*}{C}{}^{j\nu\sigma}\overset{*}{g}{}^{i\mu}=0,\\&\qquad(\mu,\nu,\sigma=0,1,2,3,4),\quad(i,j=0,1,2,3).\end{aligned} \tag{3.15}$$

4. Conclusion and Discusson

The equation (3.14) is the necessary and sufficient condition that plane sound waves could exist in the form of finite amplitude and stationary state while the lefthand side of equation (3.15) represents the deviation from the plane sound waves, as is seen from the process of derivation. The equation (3.15) does not generally hold unless the coefficients of connection $C^{\mu\lambda\varepsilon}$ vanish. Therefore, we see that the vanishing of all the coefficient of connection together with (3.14) is the sufficient condition that the plane sound waves in finite amplitude can propergate in a compressible invicid fluid without distorsion.

Thus it will be recognized that our coefficients of connection is regarded as the deviation from the plane sound waves in the above sense. In other words our coefficients of conection correspond to distorsion of the plane sound waves.

Geometrically the vanishing of $C^{\mu\lambda\varepsilon}$ leads to the condition that our Cartan space becomes locally Euclidean since from (2.5)

$$\frac{\partial g^{\mu\lambda}}{\partial u_\varepsilon}=0\ \text{.}$$

The Earnshaw-type relation, which is derived as the condition for the existance of plane sound waves is sometimes adopted in the approximation theory of compressible fluid motion to linearize the differential equations concerned.* In the geometrical viewpoint therefore, this means that in this case we could approximate our Cartan space by its tangent Euclidean space.

References

(1) T.Uehara:RAAG Research Notes, Series 3, No.11 (1959)
(2) Y.Iri:RAAG Research Notes, Series 3,No.55 (1962)
(3) T.Uehara, Y.Iri and M.Iri: International Journal of Engineeering Sciences, 1, 497 (1962)
(4) É.Cartan:Les Espaces Métriques fondés sur la Notion d'Aire, Hermann, Paris (1933)
(5) G.Birkhoff: Hydrodynamics-A Study in Logic, Fact and Similitude, Princeton Uuiversity Press, Princeton (1950)
(6) Th. von Kármán: Journal of Aeronautical Sciences, 8, 337 (1939)
(7) H.S.Tsien: Journal of Aeronautical Siences, 6, 399 (1939)
(8) A.Busemann: Zeitschrift für angewandte Mathematik und Mechanik, 17, 73 (1937)

* See, for example, (6), (7) and (8)

圧縮性流体流れに関連する Cartan 空間の接続係数の物理的意味について

上原　武幸

工学部　機械工学科

既に圧縮性流体の運動の一般理論を面積概念に基く，Cartan 空間の幾何学によって展開することが数年前に試みられ種々の興味ある結果が得られたが，背景空間として採用されたこの空間の2種の接続係数の物理的意義はまだ明確に摑まれないで残されている。このうち面素方向に関係する接続係数の物理的意味について考察し，それが圧縮性流体中を一般の状態方程式の支配の下で伝播する平面状波面のゆがみに対応することを明らかにした。一方に於て圧縮性流体の非線型方程式を近似的に解くための1つの方法として状態方程式をいわゆる Earnshaw 型に仮定する手法があるが，これをこの結論から見れば幾何学的には Cartan 空間をその一点における接 Euclid 空間で置換えられる場合，物理的には平面波の存在を仮定できるような場合に成立する近似であるということができよう。

山形大学紀要（工学）第9巻第2号 正誤表

頁	行	誤	正
22	下から6行目	$F=\frac{\pi}{4}GR^4\theta^3$	$F=\frac{\pi}{4}GR^4\theta^2$
〃	〃		式の次に (1) と入れる
31～41	見　出　し	(Ⅲ)	(Ⅱ)
72	下から7行目	$\nabla^2\cdot B$	$\nabla^2 B$
73	上から5行目	函　数	関　数
76	上から11行目	$\left(\frac{1}{2}ir_{+-1}\right)$	$\left(\frac{1}{2}ir_+\right)$
〃	上から13行目	$r_\pm=\mathrm{T_{an}}^{-1}$	$r_\pm=\mathrm{Tan}^{-1}$
77	上から12行目	(4・21)	(4・2・1)
〃	下から7行目	$e^{-i\theta j_z}$	$e^{-i\theta_{j_z}}$
〃	下から5行目	$e^{-i\theta}\theta_{EZ}$	$e^{-i\theta_{E_z}}$
〃	下から2行目	θj_Z 及び θ_{EZ}	θ_{j_z} 及び θ_{E_z}
〃	〃	の位相か	の位相が
82	下から1行目	の位相と x との	の位相と χ との
94	上から9行目	$\xi>\xi$	$\xi>\bar{\xi}$
101	第　20　図	R_2	R_{a2}
〃	下から17行目	$\sqrt{R_a^2-R_q^2}$	$\sqrt{R_a^2-R_g^2}$
〃	下から16行目	$\sqrt{(R_q+\delta)^2-R_q^2}$	$\sqrt{(R_g+\delta)^2-R_g^2}$
〃	下から15行目	R_q	R_g
105	下から12行目	低	位
〃	下から5行目	$l$sinx	$l\sin\alpha$
106	上から5行目	D_q	D_g
118	第　3　図		タテジク　Rolling Force Pkg/mm
	第　4　図		横ジク　H_RB
130	上から13行目	不鮮明	[B] C_1 の
〃	上から14行目	〃	磁心 C_1 が点
〃	下から2行目	〃	$\int_{\frac{\pi+\theta_1}{w}}^{\frac{2\pi}{w}}$
131	下から13行目	〃	V'_{D4},　r'_{D1}
153～161	見　出　し	近野：	近野・富川・高野：
153	下から13行目	$\sqrt{\left\{\left(\frac{1}{w_1^2}-\frac{1}{w_2^2}+\right.\right.}$	$\sqrt{\left\{\left(\frac{1}{w_1^2}-\frac{1}{w_2^2}\right)+\right.}$
155	下から1行目 (注)	$\left.\frac{1}{L_2}\left(\frac{w_1}{w_2}\right)\right\}$	$\left.\frac{1}{L_2}\left(\frac{w_1}{w_2}\right)\right\}^2$
157	上から2行目	らの側定を	らの測定を
167	下から11行目	Sc= の式	(13)
〃	下から10行目	Sc≒ の式	(14)
169	下から1行目	$\frac{w^2-w_f^{20}}{w^2-w_{f\infty}^2}$	$\frac{w-w_{f0}^2}{w^2-w_{f\infty}^2}$
190	下から5行目	Ou	OV
199	上から2行目	Frectional	Fractional
207	下から8行目	explicitely	explicitly
〃	下から1行目	$(\imath,j=0,1,2,3)$	$(i,j=0,1,2,3)$
208	上から10行目	$-\overset{※}{C}{}^{iv\sigma}\overset{※}{g}{}^{i\mu}$	$-\overset{※}{C}{}^{jv\sigma}\overset{※}{g}{}^{i\mu}$
〃	上から17行目	$\overset{※}{G}{}^{ii\mu\nu}=0$	$\overset{※}{G}{}^{ij\mu\nu}=0$
〃	上から21行目	Discusson	Discussion
211	下から17行目	electtrode	electrode
〃	下から7行目	Inor der	In order
213～217	見　出　し	SUGANVMA	SUGANUMA
215	下から1行目	craked	cracked
216	下から14行目	NO_{3-}	NO_3^-